

# Deming regression

MethComp package

May 2007

Anders Christian Jensen Steno Diabetes Center, Gentofte, Denmark  
acjs@steno.dk

## Contents

1	Introduction	1
2	Deming regression	1
3	The likelihood function	1
4	Solving for $\xi_i$	2
5	Solving for $\alpha$	2
6	Solving for $\beta$	3
7	Solving for $\xi_i$ - again	6
8	Solving for $\sigma^2$	6
9	Summing up	7
10	The Deming function	8

## 1 Introduction

This document is related to the `Deming` function in the package `MethComp` and contains the derivation of the maximum likelihood estimates related to the Deming regression model. It is based on the book 'Models in regression and related topics' (chapter three), from 1969 by Peter Sprent, but with more detailed calculations included.

## 2 Deming regression

The mathematical model  $\eta = \alpha + \beta\xi$  describes a linear relationship between two variables  $\xi$  and  $\eta$ . Observations  $x$  and  $y$  of two variables are usually described by a regression of  $y$  on  $x$  where  $x$  is assumed to be observed without error (or, equivalently using the conditional distribution of  $y$  given  $x$ ). In linear regression with observations subject to additive random variation on both  $x$  and  $y$  and observed values for individuals  $(x_i, y_i), i = 1, \dots, n$ , a model may be written

$$x_i = \xi_i + e_{xi},$$

$$y_i = \eta_i + e_{yi} = \alpha + \beta\xi_i + e_{yi},$$

where  $e_{xi}$  and  $e_{yi}$  denotes the random part of the model. This is known as a functional relationship because the  $\xi_i$ 's are assumed to be fixed parameters, as opposed to a structural relationship where some distribution for the  $\xi_i$ 's is assumed. In the following it is assumed that the  $e_{xi}$ s are iid with  $e_{yi} \sim N(0, \sigma^2)$ , and that the  $e_{yi}$ s are iid with  $e_{yi} \sim N(0, \lambda\sigma^2)$ , for some  $\lambda > 0$ . Furthermore  $e_{xi}$  is assumed to be independent of  $e_{yi}$ .

The aim of this document is to derive the maximum likelihood estimates for  $\alpha, \beta, \xi_i$  and  $\sigma^2$  in the functional model stated above.

## 3 The likelihood function

The likelihood function  $f_{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n}(\alpha, \beta, \xi_1, \xi_2, \dots, \xi_n, \sigma^2)$  denoted  $f$  is

$$f = \prod_{i=1}^n (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left(-\frac{(x_i - \xi_i)^2}{2\sigma^2}\right) (2\pi\lambda\sigma^2)^{-\frac{1}{2}} \exp\left(-\frac{(y_i - \alpha - \beta\xi_i)^2}{2\lambda\sigma^2}\right)$$

and the loglikelihood, denoted  $L$ , is

$$\begin{aligned} L &= \sum_{i=1}^n -\frac{1}{2} \log(2\pi\sigma^2) - \frac{(x_i - \xi_i)^2}{2\sigma^2} - \frac{1}{2} \log(2\pi\lambda\sigma^2) - \frac{(y_i - \alpha - \beta\xi_i)^2}{2\lambda\sigma^2} \\ &= -\frac{n}{2} \log(4\pi^2) - \frac{n}{2} \log(\lambda\sigma^4) - \frac{\sum_{i=1}^n (x_i - \xi_i)^2}{2\sigma^2} - \frac{\sum_{i=1}^n (y_i - \alpha - \beta\xi_i)^2}{2\lambda\sigma^2}. \end{aligned}$$

It follows that the likelihood function is not bounded from above when  $\sigma^2$  goes to 0, so in the following it is assumed that  $\sigma^2 > 0$ .

## 4 Solving for $\xi_i$

Differentiation of  $L$  with respect to  $\xi_i$  gives

$$\begin{aligned}\frac{\partial L}{\partial \xi_i} &= \frac{\partial}{\partial \xi_i} \left( -\frac{\sum_{i=1}^n (x_i - \xi_i)^2}{2\sigma^2} - \frac{\sum_{i=1}^n (y_i - \alpha - \beta \xi_i)^2}{2\lambda\sigma^2} \right) \\ &= \frac{(x_i - \xi_i)}{\sigma^2} + \frac{\beta(y_i - \alpha - \beta \xi_i)}{\lambda\sigma^2}.\end{aligned}$$

Setting  $\frac{\partial L}{\partial \xi_i}$  equal to zero yields

$$\frac{\partial L}{\partial \xi_i} = 0 \Rightarrow \xi_i = \frac{\lambda\sigma^2 x_i + \beta\sigma^2 y_i - \beta\alpha\sigma^2}{\lambda\sigma^2 + \beta^2\sigma^2} = \frac{\lambda x_i + \beta(y_i - \alpha)}{\lambda + \beta^2}. \quad (1)$$

So to estimate  $\xi_i$ , estimates for  $\beta$  and  $\alpha$  are needed. Therefore focus is turned to the derivation of  $\hat{\alpha}$ .

## 5 Solving for $\alpha$

Differentiation of  $L$  with respect to  $\alpha$  gives

$$\begin{aligned}\frac{\partial L}{\partial \alpha} &= \frac{\partial}{\partial \alpha} \left( -\frac{\sum_{i=1}^n (y_i - \alpha - \beta \xi_i)^2}{2\lambda\sigma^2} \right) \\ &= \frac{\sum_{i=1}^n (y_i - \alpha - \beta \xi_i)}{\lambda\sigma^2},\end{aligned}$$

and putting  $\frac{\partial L}{\partial \alpha}$  equal to zero yields

$$\frac{\partial L}{\partial \alpha} = 0 \Rightarrow \alpha = \frac{1}{n} \sum_{i=1}^n (y_i - \beta \xi_i).$$

Now one can use (1) to dispense with  $\xi_i$

$$\begin{aligned}
\alpha &= \frac{1}{n} \sum_{i=1}^n (y_i - \beta \xi_i) \\
&= \frac{1}{n} \sum_{i=1}^n \left( y_i - \beta \frac{\lambda x_i + \beta (y_i - \alpha)}{\lambda + \beta^2} \right) \\
&= \frac{1}{n} \sum_{i=1}^n \left( y_i - \beta \frac{\lambda x_i + \beta y_i}{\lambda + \beta^2} + \frac{\beta^2 \alpha}{\lambda + \beta^2} \right) \\
&\Downarrow \\
\alpha \left( 1 - \frac{\beta^2}{\lambda + \beta^2} \right) &= \frac{1}{n} \sum_{i=1}^n \left( y_i - \beta \frac{\lambda x_i + \beta y_i}{\lambda + \beta^2} \right) \\
&= \frac{1}{n} \sum_{i=1}^n \left( y_i \left( 1 - \frac{\beta^2}{\lambda + \beta^2} \right) - x_i \frac{\beta \lambda}{\lambda + \beta^2} \right) \\
&\Downarrow \\
\alpha &= \frac{1}{n} \sum_{i=1}^n \left( y_i - x_i \frac{\beta \lambda}{\lambda + \beta^2} \frac{\lambda + \beta^2}{\lambda} \right) \\
&= \frac{1}{n} \sum_{i=1}^n (y_i - x_i \beta) \\
&= \bar{y} - \bar{x} \beta.
\end{aligned}$$

Hence the estimate for  $\alpha$  becomes

$$\hat{\alpha} = \bar{y} - \bar{x} \hat{\beta}.$$

## 6 Solving for $\beta$

Differentiation of  $L$  with respect to  $\beta$  gives

$$\frac{\partial L}{\partial \beta} = \frac{\partial}{\partial \beta} \left( -\frac{\sum_{i=1}^n (y_i - \alpha - \beta \xi_i)^2}{2\lambda \sigma^2} \right) = \frac{\sum_{i=1}^n (y_i - \alpha - \beta \xi_i) \xi_i}{\lambda \sigma^2}.$$

Setting  $\frac{\partial L}{\partial \beta}$  equal to zero yields

$$\frac{\partial L}{\partial \beta} = 0 \Leftrightarrow \sum_{i=1}^n (y_i - \alpha - \beta \xi_i) \xi_i = 0,$$

and using (1)

$$\begin{aligned} 0 &= \sum_{i=1}^n (y_i - \alpha - \beta \xi_i) \xi_i \\ &= \sum_{i=1}^n \left( y_i - \alpha - \beta \frac{\lambda x_i + \beta(y_i - \alpha)}{\lambda + \beta^2} \right) \frac{\lambda x_i + \beta(y_i - \alpha)}{\lambda + \beta^2}. \end{aligned}$$

This implies that

$$\begin{aligned} 0 &= \sum_{i=1}^n \left( (y_i - \alpha)(\lambda + \beta^2) - \beta \lambda x_i - \beta^2(y_i - \alpha) \right) (\lambda x_i + \beta(y_i - \alpha)) \\ &= \sum_{i=1}^n \lambda^2 x_i (y_i - \alpha) + \beta^2 \lambda x_i (y_i - \alpha) - \beta \lambda^2 x_i^2 - \beta^2 \lambda x_i (y_i - \alpha) + \\ &\quad \sum_{i=1}^n \beta \lambda (y_i - \alpha)^2 + \beta^3 \lambda (y_i - \alpha)^2 - \beta^2 \lambda x_i (y_i - \alpha) - \beta^3 (y_i - \alpha)^2 \\ &= -\beta^2 \lambda \left( \sum_{i=1}^n x_i y_i \right) - \beta \lambda^2 \left( \sum_{i=1}^n x_i^2 \right) + \lambda^2 \left( \sum_{i=1}^n x_i y_i \right) \\ &\quad + \beta^2 \lambda \alpha \left( \sum_{i=1}^n x_i \right) + \beta \lambda \left( \sum_{i=1}^n (y_i - \alpha)^2 \right) - \lambda^2 \alpha \left( \sum_{i=1}^n x_i \right). \end{aligned}$$

Dividing with  $\lambda$  and using the fact that  $\alpha = \bar{y} - \bar{x} \cdot \beta$  it is seen that

$$\begin{aligned} 0 &= -\beta^2 \left( \sum_{i=1}^n x_i y_i \right) - \beta \lambda \left( \sum_{i=1}^n x_i^2 \right) + \lambda \left( \sum_{i=1}^n x_i y_i \right) + \beta^2 (\bar{y} - \bar{x} \cdot \beta) \left( \sum_{i=1}^n x_i \right) \\ &\quad + \beta \left( \sum_{i=1}^n (y_i - (\bar{y} - \bar{x} \cdot \beta))^2 \right) - \lambda (\bar{y} - \bar{x} \cdot \beta) \left( \sum_{i=1}^n x_i \right) \\ &= -\beta^2 \left( \sum_{i=1}^n x_i y_i \right) - \beta \lambda \left( \sum_{i=1}^n x_i^2 \right) + \lambda \left( \sum_{i=1}^n x_i y_i \right) + \beta^2 \bar{y} \cdot \left( \sum_{i=1}^n x_i \right) \\ &\quad - \beta^3 \bar{x} \cdot \beta \left( \sum_{i=1}^n x_i \right) + \beta \left( \sum_{i=1}^n y_i^2 \right) + \beta \left( \sum_{i=1}^n (\bar{y} - \bar{x} \cdot \beta)^2 \right) \\ &\quad - 2\beta \left( \sum_{i=1}^n y_i (\bar{y} - \bar{x} \cdot \beta) \right) - \lambda \bar{y} \cdot \left( \sum_{i=1}^n x_i \right) + \lambda \bar{x} \cdot \beta \left( \sum_{i=1}^n x_i \right). \end{aligned}$$

Splitting up the sums even more gives

$$\begin{aligned}
0 &= -\beta^2 \left( \sum_{i=1}^n x_i y_i \right) - \beta \lambda \left( \sum_{i=1}^n x_i^2 \right) + \lambda \left( \sum_{i=1}^n x_i y_i \right) + \beta^2 \bar{y} \cdot \left( \sum_{i=1}^n x_i \right) - \beta^3 \bar{x} \cdot \beta \left( \sum_{i=1}^n x_i \right) \\
&+ \beta \left( \sum_{i=1}^n y_i^2 \right) + \beta \left( \sum_{i=1}^n \bar{y} \cdot^2 \right) + \beta \left( \sum_{i=1}^n (\bar{x} \cdot \beta)^2 \right) - 2\beta \left( \sum_{i=1}^n \bar{y} \cdot \bar{x} \cdot \beta \right) - 2\beta \left( \sum_{i=1}^n y_i \bar{y} \cdot \right) \\
&+ 2\beta \left( \sum_{i=1}^n y_i \bar{x} \cdot \beta \right) - \lambda \bar{y} \cdot \left( \sum_{i=1}^n x_i \right) + \lambda \bar{x} \cdot \beta \left( \sum_{i=1}^n x_i \right).
\end{aligned}$$

Finally the terms are sorted and collected according to powers of  $\beta$ :

$$\begin{aligned}
0 &= \beta^3 \left( \sum_{i=1}^n \bar{x} \cdot^2 - \bar{x} \cdot \sum_{i=1}^n x_i \right) \\
&+ \beta^2 \left( \bar{y} \cdot \sum_{i=1}^n x_i - \sum_{i=1}^n x_i y_i - 2 \sum_{i=1}^n \bar{y} \cdot \bar{x} \cdot + 2 \sum_{i=1}^n y_i \bar{x} \cdot \right) \\
&+ \beta \left( \sum_{i=1}^n y_i^2 - \lambda \sum_{i=1}^n x_i^2 + \sum_{i=1}^n \bar{y} \cdot^2 - 2 \sum_{i=1}^n y_i \bar{y} \cdot + \lambda \bar{x} \cdot \sum_{i=1}^n x_i \right) \\
&+ \lambda \left( \sum_{i=1}^n x_i y_i - \bar{y} \cdot \sum_{i=1}^n x_i \right).
\end{aligned}$$

Since

- $\sum_{i=1}^n \bar{x} \cdot^2 - \bar{x} \cdot \sum_{i=1}^n x_i = 0$
- $\bar{y} \cdot \sum_{i=1}^n x_i - \sum_{i=1}^n x_i y_i - 2 \sum_{i=1}^n \bar{y} \cdot \bar{x} \cdot + 2 \sum_{i=1}^n y_i \bar{x} \cdot = -\text{SPD}_{xy}$
- $\sum_{i=1}^n y_i^2 - \lambda \sum_{i=1}^n x_i^2 + \sum_{i=1}^n \bar{y} \cdot^2 - 2 \sum_{i=1}^n y_i \bar{y} \cdot + \lambda \bar{x} \cdot \sum_{i=1}^n x_i = \text{SSD}_y - \lambda \text{SSD}_x$
- $\sum_{i=1}^n x_i y_i - \bar{y} \cdot \sum_{i=1}^n x_i = \text{SPD}_{xy}$

it is clear that the derivation of  $\beta$  comes down to solve

$$-\beta^2 \text{SPD}_{xy} + \beta (\text{SSD}_y - \lambda \text{SSD}_x) + \lambda \text{SPD}_{xy} = 0. \quad (2)$$

For  $\text{SPD}_{xy} \neq 0$  this implies that

$$\begin{aligned}
\beta &= \frac{-(\text{SSD}_y - \lambda \text{SSD}_x) \pm \sqrt{(\text{SSD}_y - \lambda \text{SSD}_x)^2 - 4(-\text{SPD}_{xy})\lambda \text{SPD}_{xy}}}{-2\text{SPD}_{xy}} \\
&= \frac{\text{SSD}_y - \lambda \text{SSD}_x \pm \sqrt{(\text{SSD}_y - \lambda \text{SSD}_x)^2 + 4\lambda \text{SPD}_{xy}^2}}{2\text{SPD}_{xy}}.
\end{aligned}$$

Since  $\text{SSD}_y - \lambda \text{SSD}_x \leq \sqrt{(\text{SSD}_y - \lambda \text{SSD}_x)^2 + 4\lambda \text{SPD}_{xy}^2}$  there is always a positive and a negative solution to (2). The desired solution should always have the same sign as  $\text{SPD}_{xy}$ , hence the solution with the positive numerator is selected. Therefore

$$\hat{\beta} = \frac{\text{SSD}_y - \lambda \text{SSD}_x + \sqrt{(\text{SSD}_y - \lambda \text{SSD}_x)^2 + 4\lambda \text{SPD}_{xy}^2}}{2\text{SPD}_{xy}}.$$

## 7 Solving for $\xi_i$ - again

With estimates for  $\beta$  and  $\alpha$  it is now possible to estimate  $\xi_i$  using (1):

$$\hat{\xi}_i = \frac{\lambda x_i + \hat{\beta}(y_i - \hat{\alpha})}{\lambda + \hat{\beta}^2}.$$

## 8 Solving for $\sigma^2$

Differentiation of  $L$  with respect to  $\sigma^2$  gives

$$\begin{aligned} \frac{\partial L}{\partial \sigma^2} &= \frac{\partial}{\partial \sigma^2} \left( -\frac{n}{2} \log(\lambda \sigma^4) - \frac{\sum_{i=1}^n (x_i - x_{i_i})^2}{2\sigma^2} - \frac{\sum_{i=1}^n (y_i - \alpha - \beta \xi_i)^2}{2\lambda \sigma^2} \right) \\ &= \frac{-n\sigma^2}{\sigma^4} + \frac{\sum_{i=1}^n (x_i - x_{i_i})^2}{2\sigma^4} + \frac{\sum_{i=1}^n (y_i - \alpha - \beta \xi_i)^2}{2\lambda \sigma^4} \\ &= \frac{-2\lambda n \sigma^2 + \lambda \sum_{i=1}^n (x_i - x_{i_i})^2 + \sum_{i=1}^n (y_i - \alpha - \beta \xi_i)^2}{2\lambda \sigma^4}, \end{aligned}$$

and setting  $\frac{\partial L}{\partial \sigma^2}$  equal to zero yields

$$\begin{aligned} \frac{\partial L}{\partial \sigma^2} = 0 &\Rightarrow -2\lambda n \sigma^2 + \lambda \sum_{i=1}^n (x_i - x_{i_i})^2 + \sum_{i=1}^n (y_i - \alpha - \beta \xi_i)^2 = 0 \\ &\Rightarrow \sigma^2 = \frac{\lambda \sum_{i=1}^n (x_i - \xi_i)^2 + \sum_{i=1}^n (y_i - \alpha - \beta \xi_i)^2}{2\lambda n}. \end{aligned}$$

To get a central estimate of  $\sigma^2$  one must divide by  $n - 2$  instead of  $2n$  since there are  $n + 2$  parameters to be estimated, namely  $\xi_1, \xi_2, \dots, \xi_n, \alpha$  and  $\beta$ . Hence the degrees of freedom are  $2n - (n + 2) = n - 2$ . Therefore

$$\hat{\sigma}^2 = \frac{\lambda \sum_{i=1}^n (x_i - \hat{\xi}_i)^2 + \sum_{i=1}^n (y_i - \hat{\alpha} - \hat{\beta} \hat{\xi}_i)^2}{2\lambda(n - 2)}.$$



## 9 Summing up

$$\begin{aligned}\hat{\alpha} &= \bar{y} - \bar{x}\hat{\beta} \\ \hat{\beta} &= \frac{\text{SSD}_y - \lambda\text{SSD}_x + \sqrt{(\text{SSD}_y - \lambda\text{SSD}_x)^2 + 4\lambda\text{SPD}_{xy}^2}}{2\text{SPD}_{xy}} \\ \hat{\sigma} &= \sqrt{\frac{\lambda \sum_{i=1}^n (x_i - \hat{\xi}_i)^2 + \sum_{i=1}^n (y_i - \hat{\alpha} - \hat{\beta}\hat{\xi}_i)^2}{2\lambda(n-2)}} \\ \hat{\xi}_i &= \frac{\lambda x_i + \hat{\beta}(y_i - \hat{\alpha})}{\lambda + \hat{\beta}^2}\end{aligned}$$

These formula are implemented in the `Deming` function in the `MethComp` package.

## 10 The Deming function

```

Deming <-
function( x, y, vr=sdr^2, sdr=sqrt(vr), boot=FALSE, keep.boot=FALSE, alpha=0.05 )
{
  if( missing( vr ) & missing( sdr ) ) var.ratio <- 1
  else var.ratio <- vr
  vn <- c( deparse( substitute( x ) ),
           deparse( substitute( y ) ) )
  pn <- c( "Intercept", "Slope", paste( "sigma", vn, sep="." ) )

  alfa <- alpha
  dfr <- data.frame( x=x, y=y )
  dfr <- dfr[complete.cases(dfr),]
  x <- dfr$x
  y <- dfr$y
  n <- nrow( dfr )
  SSDy <- var( y )*(n-1)
  SSDx <- var( x )*(n-1)
  SPDxy <- cov( x, y )*(n-1)
  beta <- ( SSDy - var.ratio*SSDx +
            sqrt( ( SSDy - var.ratio*SSDx )^2 +
                  4*var.ratio*SPDxy^2 ) ) / ( 2*SPDxy)
  alpha <- mean( y ) - mean( x ) * beta
  ksi <- ( var.ratio*x + beta*(y-alpha) )/(var.ratio+beta^2)
  sigma.x <- ( var.ratio*sum( (x-ksi)^2 ) +
              sum( (y-alpha-beta*ksi)^2 ) ) /
            ( (n-2)*var.ratio )
  sigma.y <- var.ratio*sigma.x
  sigma.x <- sqrt( sigma.x )
  sigma.y <- sqrt( sigma.y )
  if( !boot ){
    res <- c( alpha, beta, sigma.x, sigma.y )
    names( res ) <- pn
  }
  else
  {
    if( is.numeric( boot ) ) N <- boot else N <- 1000
    res <- matrix( NA, N, 4 )
    for( i in 1:N )
    {
      wh <- sample( 1:n, n, replace=TRUE )
      res[i,] <- Deming( x[wh], y[wh], vr=var.ratio, boot=FALSE )
    }
    ests <- cbind( c(alpha,beta,sigma.x, sigma.y),
                  se <- sqrt( diag( cov( res ) ) ) ),
                  t( apply( res, 2, quantile, probs=c(0.5,alfa/2,1-alfa/2 ), na.rm=T ) ) )
    colnames( res ) <- rownames( ests ) <- pn
    colnames( ests )<- c("Estimate", "S.e.(boot)", colnames(ests)[3:5] )
    if(keep.boot)
    {
      print( ests )
      invisible( res )
    }
  }
  else
  {
    cat( vn[2], " = alpha + beta*", vn[1], "\n" )
    ests
  }
}
}

```